

Conneted sum of representations of knot groups

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Abstract

When two boundary-parabolic representations of knot groups are given, we introduce the connected sum of these representations and show several natural properties including the unique factorization property. Furthermore, the complex volume of the connected sum is the sum of each complex volumes modulo $i\pi^2$ and the twisted Alexander polynomial of the connected sum is the product of each polynomials with normalization.

1 Introduction

For any oriented knots K_1 and K_2 , the connected sum $K_1 \# K_2$ is well-defined and has many natural properties. For example, any knot can be uniquely decomposed into prime knots. Also, the simplicial volumes $\text{vol}(K_1)$, $\text{vol}(K_2)$ and $\text{vol}(K_1 \# K_2)$ of K_1 , K_2 and $K_1 \# K_2$, respectively, satisfy $\text{vol}(K_1 \# K_2) = \text{vol}(K_1) + \text{vol}(K_2)$. Furthermore, for the Alexander polynomials Δ_{K_1} , Δ_{K_2} and $\Delta_{K_1 \# K_2}$ of K_1 , K_2 and $K_1 \# K_2$, respectively, we have $\Delta_{K_1 \# K_2} = \Delta_{K_1} \cdot \Delta_{K_2}$.

On the other hand, many important invariants are defined for a boundary-parabolic representation $\rho : \pi_1(K) \rightarrow \text{PSL}(2, \mathbb{C})$ and its lift $\tilde{\rho} : \pi_1(K) \rightarrow \text{SL}(2, \mathbb{C})$ of the knot group $\pi_1(K)$, where *the knot group* is the fundamental group of the knot complement $\mathbb{S}^3 \setminus K$ and the *boundary-parabolic*¹ means any meridian loop of the boundary-torus maps to a parabolic element in $\text{PSL}(2, \mathbb{C})$ under ρ . For example, the complex volume $\text{vol}(\rho) + i \text{cs}(\rho)$ and the twisted Alexander polynomial $\Delta_{K, \tilde{\rho}}$ are some of the important invariants.

For two boundary-parabolic representations $\rho_1 : \pi_1(K_1) \rightarrow \text{PSL}(2, \mathbb{C})$ and $\rho_2 : \pi_1(K_2) \rightarrow \text{PSL}(2, \mathbb{C})$, we will define the connected sum of ρ_1 and ρ_2

$$\rho_1 \# \rho_2 : \pi_1(K_1 \# K_2) \rightarrow \text{PSL}(2, \mathbb{C})$$

in Section 2. (Note: After the publication of this article, serious errors were found. The author wanted to preserve the content of the publication, so he added the errata in the appendix.) Then this definition satisfies the unique factorization property; for any oriented

¹Boundary-parabolic representation is also called *parabolic representation* in many other texts.

knot $K = K_1 \# \dots \# K_g$ and any boundary-parabolic representation $\rho : \pi_1(K) \rightarrow \mathrm{PSL}(2, \mathbb{C})$, there exist unique boundary-parabolic representations

$$\rho_j : \pi_1(K_j) \rightarrow \mathrm{PSL}(2, \mathbb{C}) \quad (j = 1, \dots, g)$$

satisfying $\rho = \rho_1 \# \dots \# \rho_g$ up to conjugate. (If two same knots K_j and K_k appear in K , then the indices of ρ_j and ρ_k can be exchanged.)

Using this definition, we will show the following additivity of complex volumes

$$\mathrm{vol}(\rho_1 \# \rho_2) + i \mathrm{cs}(\rho_1 \# \rho_2) \equiv (\mathrm{vol}(\rho_1) + i \mathrm{cs}(\rho_1)) + (\mathrm{vol}(\rho_2) + i \mathrm{cs}(\rho_2)) \pmod{i\pi^2}, \quad (1)$$

in Section 3. The author believes (1) was already known to some experts because the knot complement $\mathbb{S}^3 \setminus (K_1 \# K_2 \cup \{\text{two points}\})$ is obtained by gluing $\mathbb{S}^3 \setminus (K_1 \cup \{\text{two points}\})$ and $\mathbb{S}^3 \setminus (K_2 \cup \{\text{two points}\})$ along $\mathbb{T}^2 \setminus \{\text{two points}\}$, a torus minus two points.² However, the proof in Section 3 will be combinatorial and very simple. Furthermore, while proving (1), we will show the solutions of the hyperbolicity equations \mathcal{I}_1 and \mathcal{I}_2 , which correspond to the five-term triangulations of $\mathbb{S}^3 \setminus (K_1 \cup \{\text{two points}\})$ and $\mathbb{S}^3 \setminus (K_2 \cup \{\text{two points}\})$, respectively, are determined by the solution of \mathcal{I} , which corresponds to the triangulation of $\mathbb{S}^3 \setminus (K_1 \# K_2 \cup \{\text{two points}\})$. (See Lemma 3.4.) This is not a usual situation because, in general, if we glue two manifolds, then the set of the hyperbolicity equations changes, and even small change on the equations induces radical change on the solutions. Therefore, the solution of the glued manifold usually cannot detect the solutions of the original two manifolds. However, it works for our case in Section 3 because we will use combinatorial method.

In Section 4, we will show the twisted Alexander polynomial $\Delta_{K_1 \# K_2, \widetilde{\rho}_1 \# \widetilde{\rho}_2}$ is the product of $\Delta_{K_1, \widetilde{\rho}_1}$ and $\Delta_{K_2, \widetilde{\rho}_2}$ with normalization. Finally, Section 5 will discuss an example $\rho_1 \# \rho_2 : \pi_1(3_1 \# 4_1) \rightarrow \mathrm{PSL}(2, \mathbb{C})$ and its lift $\widetilde{\rho}_1 \# \widetilde{\rho}_2 : \pi_1(3_1 \# 4_1) \rightarrow \mathrm{SL}(2, \mathbb{C})$.

Although we restrict our attention to boundary-parabolic representations for simplicity, under certain condition, all results in Section 2 and 4 are still true for general representations. It will be discussed briefly later.

Note that all representations in this article are defined up to conjugate. We follow the definition of the complex volume of a representation ρ in [7] and that of the twisted Alexander polynomial in Section 2 of [6].

2 Definition and the unique factorization

2.1 Wirtinger presentation and arc-coloring

For a fixed oriented knot diagram³ D of a knot K , let $\alpha_1, \dots, \alpha_n$ be the arcs of D . These arcs can be regarded as the meridian loops of the boundary-torus, which is expressed by small arrows in Figure 1. Then Wirtinger presentation gives a presentation of the knot group

$$\pi_1(K) = \langle \alpha_1, \dots, \alpha_n; r_1, \dots, r_{n-1} \rangle, \quad (2)$$

²The gluing map here is topologically unique because it is obtained by gluing two pairs of two vertex-oriented ideal triangles.

³ We assume D has at least one crossing.

where the relations r_1, \dots, r_n are defined in Figure 2. (We can remove one relation in $\{r_1, \dots, r_n\}$ because it can be obtained by all the others.)

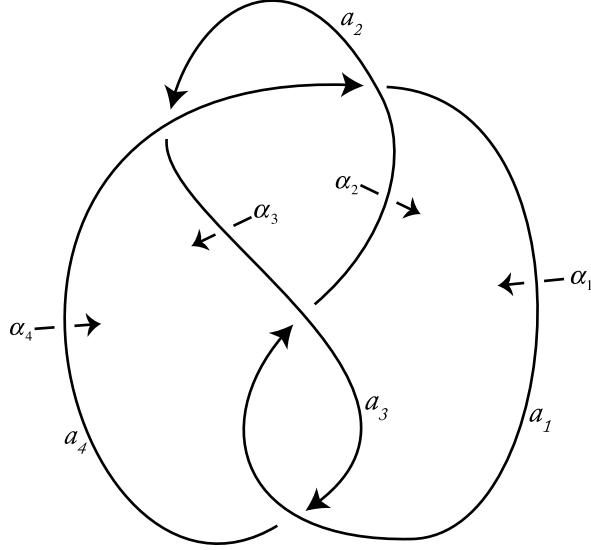


Figure 1: Knot diagram with arcs $\alpha_1, \dots, \alpha_n$ and arc-colors a_1, \dots, a_n

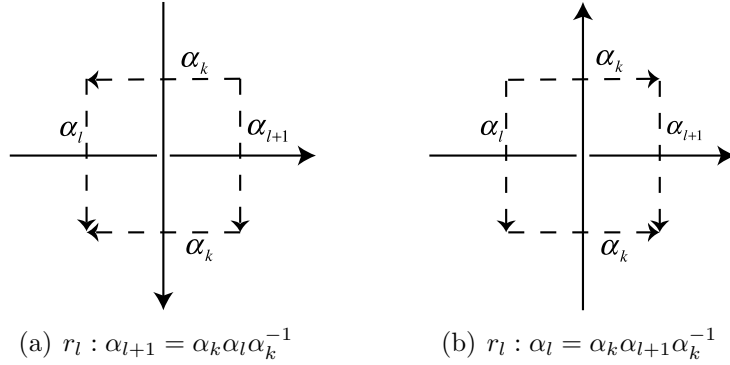


Figure 2: Relations at crossings

Let \mathcal{P} be the set of parabolic elements in $\text{PSL}(2, \mathbb{C})$. For a boundary-parabolic representation $\rho : \pi_1(K) \rightarrow \text{PSL}(2, \mathbb{C})$, put $a_k = \rho(\alpha_k) \in \mathcal{P}$ and call a_k the arc-color of α_k (induced by ρ .) Note that, due to the Wirtinger presentation, the arc-coloring determines the representation ρ uniquely (up to conjugate.) Therefore, from now on, we express the representation ρ by using the arc-coloring of a diagram D .

For $a, b \in \mathcal{P}$, we define the operation $*$ by

$$a * b = bab^{-1} \in \text{PSL}(2, \mathbb{C}). \quad (3)$$

Then the arc-colors of a crossing satisfy the relation in Figure 3. Furthermore, the operation $*b : a \mapsto a * b$ is bijective and satisfies

$$a * a = a \text{ and } (a * b) * c = (a * c) * (b * c),$$

for any $a, b, c \in \mathcal{P}$, which implies $(\mathcal{P}, *)$ is a quandle. (See [4] or [3] for details.) We define the inverse operation $*^{-1}$ by

$$a *^{-1} c = b \iff a = b * c.$$

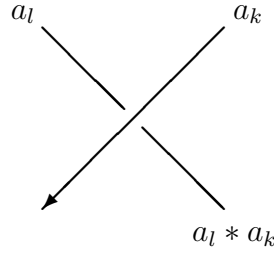


Figure 3: Arc-coloring

One trivial, but important fact is that the arc-coloring uniquely changes under the Reidemeister moves. (This is trivial because arc-coloring is uniquely determined by the representation ρ . Another way to see this fact is to consider the relationship between the Reidemeister moves and the axioms of quandle. See Figure 4.)

Definition 2.1. Let K_1 and K_2 be oriented knots with diagrams D_1 and D_2 , respectively. For $j = 1, 2$, let $\rho_j : \pi_1(K_j) \rightarrow \text{PSL}(2, \mathbb{C})$ be a boundary-parabolic representation. For the arc-colorings of D_1 and D_2 (induced by ρ_1 and ρ_2 , respectively), we make one arc-color of D_1 and another arc-color of D_2 coincided by conjugation. We denote the coincided arc-color by $a \in \mathcal{P}$. Then we define the arc-coloring of $D_1 \# D_2$ following Figure 5. The boundary-parabolic representation induced by this arc-coloring is denoted by

$$\rho_1 \# \rho_2 : \pi_1(K_1 \# K_2) \rightarrow \text{PSL}(2, \mathbb{C})$$

and is called **the connected sum of ρ_1 and ρ_2** .

Theorem 2.2. The connected sum $\rho_1 \# \rho_2$ is well-defined up to conjugate.

Proof. At first, note that the well-definedness of $K_1 \# K_2$ (up to isotopy) is already proved in standard textbooks.

Let $a \in \mathcal{P}$ be the coincided arc-color in the definition. For another arc-color $b \in \mathcal{P}$ of D_2 , there exists unique $c \in \mathcal{P}$ such that $b * c = a$. We will show the arc-coloring of the right-hand side of Figure 6 is conjugate with that of Figure 5. (In Figure 6, $D_2 * c$ means the arc-coloring of D_2 obtained by acting $*c$ to all arc-colors.)

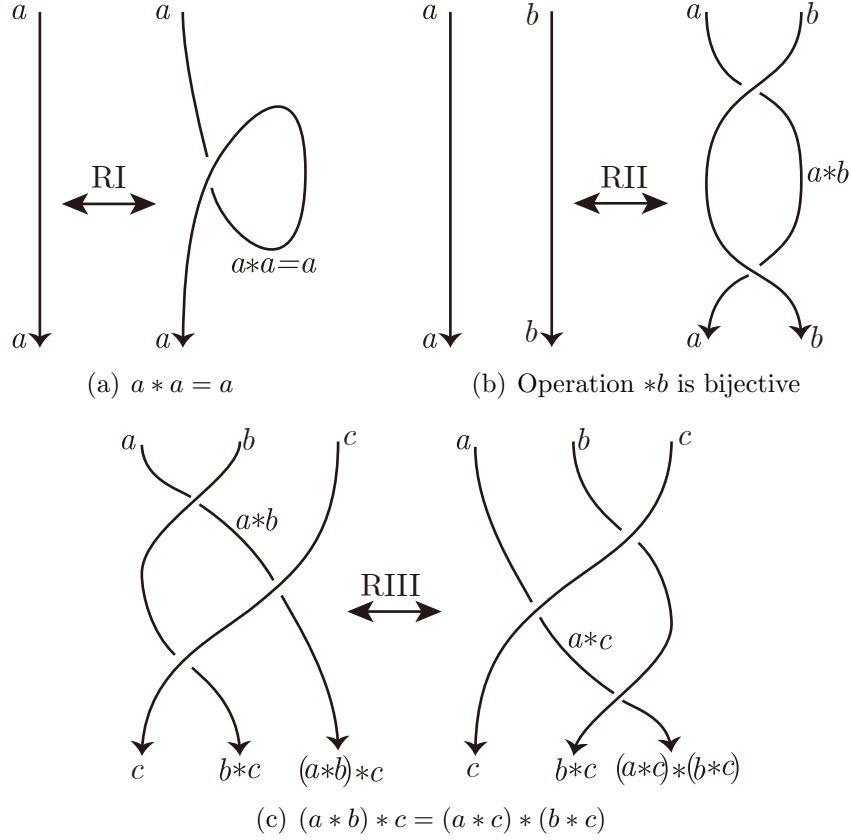


Figure 4: Reidemeister moves and the axioms of quandle

To show the coincidence, we need the observation on the changes of arc-colors in Figure 7. The observation shows that the arc-colors outside D or $D * x$ does not change by moving D or $D * x$ across the crossing. Also note that the arc-colors of the two open arcs of D or $D * x$ are always the same.

Moving the diagram D_1 of the right-hand side of Figure 6 (or the left-hand side of Figure 8) inside $D_2 * c$, we obtain the middle picture of Figure 8. (The changed arc-color of D_1 is determined by the arc-color $a * c$ of the two arcs.) By acting $*^{-1}c$ to all arc-colors, we obtain the right-hand side of Figure 5, and the coincidence of the arc-colors is proved.

On the other hand, the arc-colorings changed by applying Reidemeister moves to the diagrams D_1 and D_2 are uniquely determined. (See Figure 4.) Therefore, changing diagrams does not have any impact on the definition of $\rho_1 \# \rho_2$.

□

Proposition 2.3. *For a boundary-parabolic representation $\rho : \pi_1(K_1 \# K_2) \rightarrow \text{PSL}(2, \mathbb{C})$, there exist unique $\rho_1 : \pi_1(K_1) \rightarrow \text{PSL}(2, \mathbb{C})$ and $\rho_2 : \pi_1(K_2) \rightarrow \text{PSL}(2, \mathbb{C})$ satisfying $\rho = \rho_1 \# \rho_2$ up to conjugate. (If $K_1 = K_2$, then the decomposition is not unique but $\rho_1 \# \rho_2 = \rho_2 \# \rho_1 = \rho$ up to conjugate.)*

Proof. Choose a diagram $D_1 \# D_2$ of $K_1 \# K_2$ as in Figure 9(a). Then the arc-colors $a, b \in \mathcal{P}$

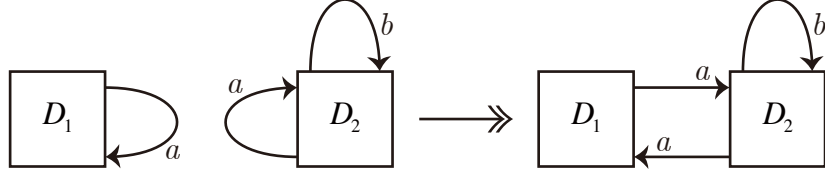


Figure 5: Arc-coloring of $D_1 \# D_2$

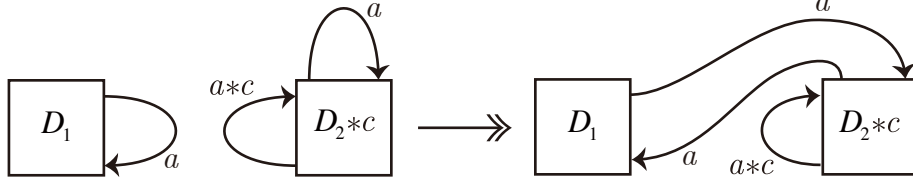


Figure 6: Arc-coloring of $D_1 \# D_2$ obtained by connecting different arcs

should satisfy $a = b$ because the corresponding meridian loops are homotopic. Hence we can define ρ_1 and ρ_2 using the arc-colorings in Figure 9(b).

To show the uniqueness, assume $\rho'_1 \# \rho'_2 = \rho = \rho_1 \# \rho_2$ up to conjugate. Then $\rho'_1 \# \rho'_2$ also induces an arc-coloring of $D_1 \# D_2$, which should be conjugate with the arc-coloring induced by ρ . Therefore, $\rho'_1 = \rho_1$ and $\rho'_2 = \rho_2$ up to conjugate. \square

The general case of $K = K_1 \# \dots \# K_g$ in Section 1 can be proved by Proposition 2.3 and the induction on g .

Remark that all discussions in this section can be easily generalized to any representation $\rho_j : \pi_1(K_j) \rightarrow \text{GL}(k, \mathbb{C})$. One obstruction is that, for ρ_1 and ρ_2 , $\rho_1 \# \rho_2$ is defined only when $\rho_1(\alpha)$ is conjugate with $\rho_2(\beta)$ for some meridian loops $\alpha \in \pi_1(K_1)$ and $\beta \in \pi_1(K_2)$. Also, generalization to links is possible if we specify which components are connected by the connected sum.

3 Complex volume of ρ

To calculate the complex volume of $\rho_1 \# \rho_2$ explicitly, we briefly review the shadow-coloring of [3] and the main result of [2].

We identify $\mathbb{C}^2 \setminus \{0\} / \pm$ with \mathcal{P} by

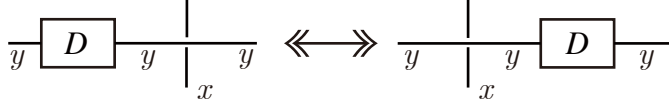
$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \longleftrightarrow \begin{pmatrix} 1 + \alpha\beta & -\alpha^2 \\ \beta^2 & 1 - \alpha\beta \end{pmatrix}. \quad (4)$$

Then the operation $*$ defined in (3) is given by

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} * \begin{pmatrix} \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} 1 + \gamma\delta & -\gamma^2 \\ \delta^2 & 1 - \gamma\delta \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \mathcal{P},$$



(a) Moving under the crossing



(b) Moving over the crossing

Figure 7: Changes of arc-colors

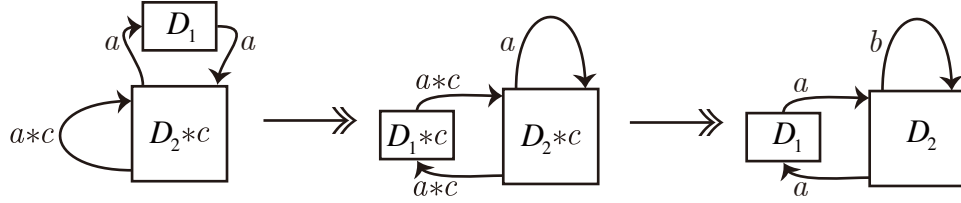


Figure 8: Coincidence of the arc-coloring

where the operation on the right-hand side is the usual matrix multiplication. The inverse operation $*^{-1}$ is given by

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} *^{-1} \begin{pmatrix} \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} 1 - \gamma\delta & \gamma^2 \\ -\delta^2 & 1 + \gamma\delta \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \mathcal{P}.$$

The Hopf map $h : \mathcal{P} \rightarrow \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ is defined by

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \mapsto \frac{\alpha}{\beta}.$$

For the given arc-coloring of the diagram D with arc-colors a_1, \dots, a_n , we assign *region-colors* $s_1, \dots, s_m \in \mathcal{P}$ to regions of D satisfying the rule in Figure 10. Note that, if an arc-coloring is fixed, then a choice of one region-color determines all the other region-colors.

Lemma 3.1. *Consider the arc-coloring induced by the boundary-parabolic representation $\rho : \pi_1(K) \rightarrow \text{PSL}(2, \mathbb{C})$. Then, for any triple $(a_k, s, s * a_k)$ of an arc-color a_k and its surrounding region-colors $s, s * a_k$ as in Figure 10, there exists a region-coloring satisfying*

$$h(a_k) \neq h(s) \neq h(s * a_k) \neq h(a_k).$$

Proof. See Proof of Lemma 2.4 in [3].

□

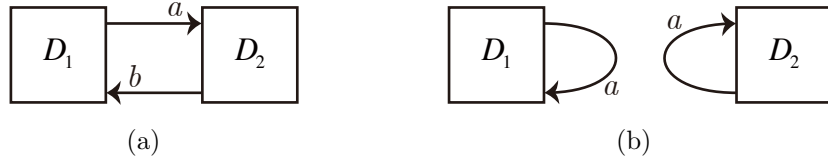


Figure 9: Arc-colors of D_1 and D_2 induced by the arc-color of $D_1 \# D_2$

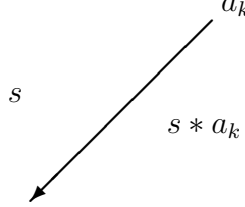


Figure 10: Region-coloring

The arc-coloring induced by ρ together with the region-coloring satisfying Lemma 3.1 is called *the shadow-coloring induced by ρ* . We choose $p \in \mathcal{P}$ so that

$$h(p) \notin \{h(a_1), \dots, h(a_n), h(s_1), \dots, h(s_m)\}. \quad (5)$$

From now on, we fix the representatives of shadow-colors in $\mathbb{C}^2 \setminus \{0\}$, not in \mathcal{P} . Note that this may cause inconsistency of some signs of arc-colors under the operation $*$. (In other words, for arc-colors $a_j, a_k, a_l \in \mathcal{P}$ with $a_j = a_k * a_l$, we allow $a_j = \pm a_k * a_l \in \mathbb{C}^2 \setminus \{0\}$. As discussed in [3], this inconsistency does not make any problem.) For $a = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$ and $b = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$ in $\mathbb{C}^2 \setminus \{0\}$, we define *the determinant* $\det(a, b)$ by

$$\det(a, b) := \det \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix} = \alpha_1 \beta_2 - \beta_1 \alpha_2.$$

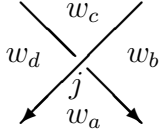
For the knot diagram D , we assign variables w_1, \dots, w_m to the regions with region-colors s_1, \dots, s_m , respectively, and define a potential function of a crossing j as in Figure 11, where $\text{Li}_2(z) = -\int_0^z \frac{\log(1-t)}{t} dt$ is the dilogarithm function.

Then the potential function of D is defined by

$$W(w_1, \dots, w_m) := \sum_{j : \text{crossings}} W_j,$$

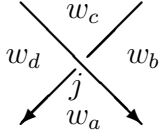
and we modify it to

$$W_0(w_1, \dots, w_m) := W(w_1, \dots, w_m) - \sum_{k=1}^m \left(w_k \frac{\partial W}{\partial w_k} \right) \log w_k.$$



$$\longrightarrow W_j := -\text{Li}_2\left(\frac{w_c}{w_b}\right) - \text{Li}_2\left(\frac{w_c}{w_d}\right) + \text{Li}_2\left(\frac{w_a w_c}{w_b w_d}\right) + \text{Li}_2\left(\frac{w_b}{w_a}\right) + \text{Li}_2\left(\frac{w_d}{w_a}\right) - \frac{\pi^2}{6} + \log \frac{w_b}{w_a} \log \frac{w_d}{w_a}$$

(a) Positive crossing



$$\longrightarrow W_j := \text{Li}_2\left(\frac{w_c}{w_b}\right) + \text{Li}_2\left(\frac{w_c}{w_d}\right) - \text{Li}_2\left(\frac{w_a w_c}{w_b w_d}\right) - \text{Li}_2\left(\frac{w_b}{w_a}\right) - \text{Li}_2\left(\frac{w_d}{w_a}\right) + \frac{\pi^2}{6} - \log \frac{w_b}{w_a} \log \frac{w_d}{w_a}$$

(b) Negative crossing

Figure 11: Potential function of the crossing j

Also, from the potential function $W(w_1, \dots, w_m)$, we define a set of equations

$$\mathcal{I} := \left\{ \exp \left(w_k \frac{\partial W}{\partial w_k} \right) = 1 \mid k = 1, \dots, m \right\}.$$

Then, from Proposition 1.1 of [1], \mathcal{I} becomes the set of hyperbolicity equations of the five-term triangulation of $\mathbb{S}^3 \setminus (K \cup \{\text{two points}\})$. Here, hyperbolicity equations are the equations that determine the complete hyperbolic structure of the triangulation, which consist of gluing equations of edges and completeness condition. According to Yoshida's construction in Section 4.5 of [5], a solution $\mathbf{w} = (w_1, \dots, w_m)$ of \mathcal{I} determines the boundary-parabolic representation

$$\rho_{\mathbf{w}} : \pi_1(\mathbb{S}^3 \setminus (K \cup \{\text{two points}\})) = \pi_1(\mathbb{S}^3 \setminus K) \longrightarrow \text{PSL}(2, \mathbb{C}),$$

up to conjugate.

Theorem 3.2 ([2]). *For any boundary-parabolic representation $\rho : \pi_1(K) \rightarrow \text{PSL}(2, \mathbb{C})$ and any knot diagram D of K , there exists the solution $\mathbf{w}^{(0)}$ of \mathcal{I} satisfying $\rho_{\mathbf{w}^{(0)}} = \rho$, up to conjugate. Furthermore,*

$$W_0(\mathbf{w}^{(0)}) \equiv i(\text{vol}(\rho) + i \text{cs}(\rho)) \pmod{\pi^2}. \quad (6)$$

The value $\text{vol}(\rho) + i \text{cs}(\rho)$ is called **the complex volume of ρ** .

The explicit formula of $\mathbf{w}^{(0)} = (w_1^{(0)}, \dots, w_m^{(0)})$ is very simple. For a region of D with region-color s_k satisfying Lemma 3.1 and region-variable w_k , the value $w_k^{(0)}$ of the region-variable is defined by

$$w_k^{(0)} := \det(p, s_k). \quad (7)$$

Corollary 3.3. *For a boundary-parabolic representation $\rho_1 \# \rho_2 : \pi_1(K_1 \# K_2) \rightarrow \text{PSL}(2, \mathbb{C})$, we have*

$$\text{vol}(\rho_1 \# \rho_2) + i \text{cs}(\rho_1 \# \rho_2) \equiv (\text{vol}(\rho_1) + i \text{cs}(\rho_1)) + (\text{vol}(\rho_2) + i \text{cs}(\rho_2)) \pmod{i \pi^2}. \quad (8)$$

Proof. For the connected sum $K_1 \# K_2$, consider a diagram $D_1 \# D_2$ and its shadow-coloring induced by $\rho_1 \# \rho_2$. (Remark that the shadow-coloring satisfies Lemma 3.1.) By rearranging the indices, we assume $\{s_1, \dots, s_l, s_{l+1}\}$ and $\{s_l, s_{l+1}, \dots, s_m\}$ are the region-colors of D_1 and D_2 , respectively, and s_l is the region-color assigned to the unbounded region of $D_1 \# D_2$. (See Figure 12(a).)

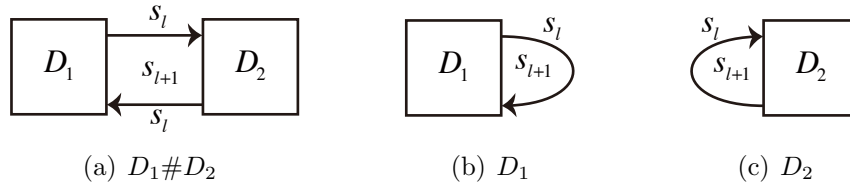


Figure 12: Region-colorings of diagrams

Let $W_1(w_1, \dots, w_l, w_{l+1})$ and $W_2(w_l, w_{l+1}, \dots, w_m)$ be the potential functions of the diagrams D_1 and D_2 in Figures 12(b) and (c), respectively. Then

$$W(w_1, \dots, w_m) = W_1(w_1, \dots, w_l, w_{l+1}) + W_2(w_l, w_{l+1}, \dots, w_m)$$

holds trivially.

Lemma 3.4. *For the solution $\mathbf{w}^{(0)} = (w_1^{(0)}, \dots, w_l^{(0)}, w_{l+1}^{(0)}, \dots, w_m^{(0)})$ of \mathcal{I} defined by (7), let $\mathbf{w}_1^{(0)} := (w_1^{(0)}, \dots, w_l^{(0)}, w_{l+1}^{(0)})$ and $\mathbf{w}_2^{(0)} := (w_l^{(0)}, w_{l+1}^{(0)}, \dots, w_m^{(0)})$. Then $\mathbf{w}_1^{(0)}$ and $\mathbf{w}_2^{(0)}$ are solutions of $\mathcal{I}_1 := \left\{ \exp\left(w_k \frac{\partial W_1}{\partial w_k}\right) = 1 \mid k = 1, \dots, l, l+1 \right\}$ and $\mathcal{I}_2 := \left\{ \exp\left(w_k \frac{\partial W_2}{\partial w_k}\right) = 1 \mid k = l, l+1, \dots, m \right\}$, respectively. Furthermore,*

$$\rho_{\mathbf{w}_j^{(0)}} = \rho_j$$

up to conjugate, and

$$(W_j)_0(\mathbf{w}_j^{(0)}) \equiv i(\text{vol}(\rho_j) + i \text{cs}(\rho_j)) \pmod{\pi^2} \quad (9)$$

for $j = 1, 2$.

Proof. Note that the arc-colorings of D_1 and D_2 induce the representations ρ_1 and ρ_2 , respectively. Both of the region-colorings $\{s_1, \dots, s_l, s_{l+1}\}$ and $\{s_l, s_{l+1}, \dots, s_m\}$ of D_1 and D_2 in Figures 12(b) and (c), respectively, satisfy Lemma 3.1. Therefore, by applying Theorem 3.2 to Figures 12(b) and (c), we obtain the results of this lemma. \square

The relation (8) is directly obtained by (6), (9) and

$$W_0(\mathbf{w}^{(0)}) = (W_1)_0(\mathbf{w}_1^{(0)}) + (W_2)_0(\mathbf{w}_2^{(0)}),$$

which complete the proof of Corollary 3.3. \square

4 Twisted Alexander polynomial of $\tilde{\rho}$

To calculate (Wada's) twisted Alexander polynomial, we briefly summarize the calculation method in Section 2 of [6].

At first, we lift the boundary-parabolic representation $\rho : \pi_1(K) \rightarrow \mathrm{PSL}(2, \mathbb{C})$ to $\tilde{\rho} : \pi_1(K) \rightarrow \mathrm{SL}(2, \mathbb{C})$ by assuming all arc-colors have trace two. As a matter of fact, this assumption was already reflected in the right-hand side of (4). Under this lifting, we can trivially obtain

$$\widetilde{\rho_1 \# \rho_2} = \tilde{\rho}_1 \# \tilde{\rho}_2.$$

Therefore, we will use $\tilde{\rho}_1 \# \tilde{\rho}_2$ instead of $\widetilde{\rho_1 \# \rho_2}$ from now on.

Consider the Wirtinger presentation of $\pi_1(K)$ in (2). Let

$$\gamma : \pi_1(K) \rightarrow \mathbb{Z} = \langle t \rangle$$

be the abelianization homomorphism given by $\gamma(\alpha_1) = \dots = \gamma(\alpha_n) = t$. We define the tensor product of $\tilde{\rho}$ and γ by

$$(\tilde{\rho} \otimes \gamma)(x) = \tilde{\rho}(x)\gamma(x),$$

for $x \in \pi_1(K)$.

From the maps $\tilde{\rho}$ and γ , we obtain natural ring homomorphisms $\tilde{\rho}_* : \mathbb{Z}[\pi_1(K)] \rightarrow M(2, \mathbb{C})$ and $\gamma_* : \mathbb{Z}[\pi_1(K)] \rightarrow \mathbb{Z}[t, t^{-1}]$, where $\mathbb{Z}[\pi_1(K)]$ is the group ring of $\pi_1(K)$ and $M(2, \mathbb{C})$ is the matrix algebra consisting of 2×2 matrices over \mathbb{C} . Combining them, we obtain a ring homomorphism

$$\tilde{\rho}_* \otimes \gamma_* : \mathbb{Z}[\pi_1(K)] \rightarrow M(2, \mathbb{C}[t, t^{-1}]).$$

Let $F_n = \langle \alpha_1, \dots, \alpha_n \rangle$ be the free group and $\psi : \mathbb{Z}[F_n] \rightarrow \mathbb{Z}[\pi_1(K)]$ be the natural surjective homomorphism. Define $\Phi : \mathbb{Z}[F_n] \rightarrow M(2, \mathbb{C}[t, t^{-1}])$ by

$$\Phi = (\tilde{\rho}_* \otimes \gamma_*) \circ \psi.$$

Consider the $(n-1) \times n$ matrix $M_{\tilde{\rho}}$ whose (k, j) -component is the 2×2 matrix

$$\Phi \left(\frac{\partial r_k}{\partial \alpha_j} \right) \in M(2, \mathbb{C}[t, t^{-1}]),$$

where $\frac{\partial}{\partial \alpha_j}$ denotes the Fox calculus. We call $M_{\tilde{\rho}}$ **the Alexander matrix associated to $\tilde{\rho}$** . We denote by $M_{\tilde{\rho}, j}$ the $(n-1) \times (n-1)$ matrix obtained from $M_{\tilde{\rho}}$ by removing the j th column for any $j = 1, \dots, n$. Then **the twisted Alexander polynomial of K associated to $\tilde{\rho}$** is defined by

$$\Delta_{K, \tilde{\rho}}(t) = \frac{\det M_{\tilde{\rho}, j}}{\det \Phi(1 - \alpha_j)}, \quad (10)$$

and it is well-defined up to t^p ($p \in \mathbb{Z}$).

If we concentrate on a boundary-parabolic representation ρ and its lift $\tilde{\rho}$, then $\det \Phi(1 - \alpha_j)$ in (10) is always $(1 - t)^2$ independent of the choice of j by the following calculation: after putting $\tilde{\rho}(\alpha_j) = P \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} P^{-1}$ for certain invertible matrix P ,

$$\det \Phi(1 - \alpha_j) = \det(P P^{-1} - t P \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} P^{-1}) = \det(1 - t \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}) = (1 - t)^2.$$

(Even when we consider a non-boundary-parabolic representation, the value of $\det \Phi(1 - \alpha_j)$ in (10) is still independent of j because all arc-colors of the knot diagram are conjugate each other.)

Now we apply this calculation method to the case of $K_1 \# K_2$ associated to $\tilde{\rho}_1 \# \tilde{\rho}_2$. For Figure 13(a), consider the Wirtinger presentation of $\pi_1(K_1)$ and $\pi_1(K_2)$ by

$$\pi_1(K_1) = \langle \alpha_1, \dots, \alpha_l \mid r_1, \dots, r_{l-1}, r_l \rangle = \langle \alpha_1, \dots, \alpha_l \mid r_1, \dots, r_{l-1} \rangle$$

and

$$\pi_1(K_2) = \langle \alpha_l, \dots, \alpha_n \mid r'_l, r_{l+1}, r_{l+2}, \dots, r_n \rangle = \langle \alpha_l, \dots, \alpha_n \mid r_{l+1}, r_{l+2}, \dots, r_n \rangle,$$

respectively. (In Figure 13, D_1 and D_2 are the diagrams of K_1 and K_2 , respectively.)

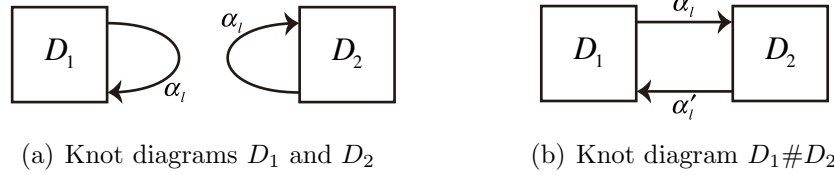


Figure 13: Knot diagrams with some arcs

Lemma 4.1. *In the above Wirtinger presentation of $\pi_1(K_1)$ and $\pi_1(K_2)$, we can present $\pi_1(K_1 \# K_2)$ by*

$$\pi_1(K_1 \# K_2) = \langle \alpha_1, \dots, \alpha_n \mid r_1, \dots, r_{l-1}, r_{l+1}, \dots, r_n \rangle.$$

Proof. In Figure 13(b), the meridian loop corresponding to α_l is homotopic to that of α'_l . Therefore, after writing down the Wirtinger presentation of $\pi_1(K_1 \# K_2)$ and substituting α'_l to α_l in all the relations, the resulting presentation is

$$\pi_1(K_1 \# K_2) = \langle \alpha_1, \dots, \alpha_n \mid r_1, \dots, r_{l-1}, r_l, r'_l, r_{l+1}, \dots, r_n \rangle. \quad (11)$$

From the fact that α_l is homotopic to α'_l , two relations in (11) are redundant, one from D_1 and another from D_2 . After removing r_l and r'_l , we complete the proof. □

Corollary 4.2. *For the boundary-parabolic representations ρ_1, ρ_2 and their lifts $\widetilde{\rho}_1, \widetilde{\rho}_2$, the twisted Alexander polynomials satisfy*

$$\Delta_{K_1 \# K_2, \widetilde{\rho}_1 \# \widetilde{\rho}_2} = (1-t)^2 \Delta_{K_1, \widetilde{\rho}_1} \Delta_{K_2, \widetilde{\rho}_2}. \quad (12)$$

Proof. Consider the Wirtinger presentations of $\pi_1(K_1)$, $\pi_1(K_2)$ and $\pi_1(K_1 \# K_2)$ above. Let M_1 be the $(l-1) \times (l-1)$ matrix whose (k, j) component is

$$\Phi \left(\frac{\partial r_k}{\partial \alpha_j} \right) (k, j = 1, \dots, l-1),$$

and M_2 be the $(n-l) \times (n-l)$ matrix whose (k, j) component is

$$\Phi \left(\frac{\partial r_k}{\partial \alpha_j} \right) (k, j = l+1, \dots, n).$$

Then

$$\begin{aligned} \Delta_{K_1 \# K_2, \widetilde{\rho}_1 \# \widetilde{\rho}_2} &= \frac{\det \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}}{\det \Phi(1 - \alpha_j)} \\ &= \det \Phi(1 - \alpha_j) \frac{\det(M_1)}{\det \Phi(1 - \alpha_j)} \frac{\det(M_2)}{\det \Phi(1 - \alpha_j)} = (1-t)^2 \Delta_{K_1, \widetilde{\rho}_1} \Delta_{K_2, \widetilde{\rho}_2}. \end{aligned}$$

□

Remark that the natural generalization of the Alexander polynomial $\Delta_K(t)$ is to define the twisted Alexander polynomial $\Delta'_{K, \widetilde{\rho}}(t)$, using different normalization from (10), by

$$\Delta'_{K, \widetilde{\rho}}(t) := \det M_{\widetilde{\rho}, j} = (1-t)^2 \Delta_{K, \widetilde{\rho}}(t).$$

Then the product formula (12) changes to

$$\Delta'_{K_1 \# K_2, \widetilde{\rho}_1 \# \widetilde{\rho}_2} = \Delta'_{K_1, \widetilde{\rho}_1} \cdot \Delta'_{K_2, \widetilde{\rho}_2},$$

which is a natural generalization of $\Delta_{K_1 \# K_2} = \Delta_{K_1} \cdot \Delta_{K_2}$.

Note that, for non-boundary-parabolic representations of oriented knots, Corollary 4.2 still holds with slight modification. The term $(1-t)^2$ in (12) should be changed to $\det \Phi(1 - \alpha_j)$, where α_j is the arc connecting two diagrams. However, as shown before, choosing any arc α_k instead of the connecting arc α_j gives the same equation $\det \Phi(1 - \alpha_k) = \det \Phi(1 - \alpha_j)$.

5 Example

For the trefoil knot 3_1 in the left-hand side and the figure-eight knot 4_1 in the right-hand side of Figure 14, we put the boundary-parabolic representation $\rho : \pi_1(3_1 \# 4_1) \rightarrow \text{PSL}(2, \mathbb{C})$

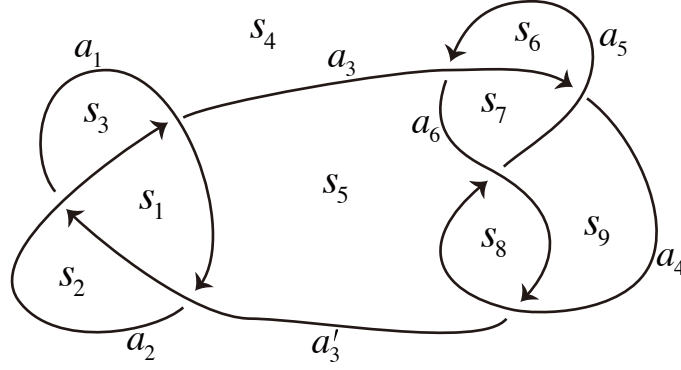


Figure 14: $3_1 \# 4_1$

determined by the arc-colors

$$a_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad a_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = a'_3,$$

$$a_4 = \begin{pmatrix} x+1 \\ x \end{pmatrix}, \quad a_5 = \begin{pmatrix} x \\ x \end{pmatrix}, \quad a_6 = \begin{pmatrix} x \\ 0 \end{pmatrix},$$

where $x = \frac{-1 \pm \sqrt{3}i}{2}$ is a solution of $x^2 + x + 1 = 0$. (We consider each arc-color a_k is assigned to the arc α_k .) Let $\rho = \rho_1 \# \rho_2$ for $\rho_j : \pi_1(K_j) \rightarrow \text{PSL}(2, \mathbb{C})$ with $j = 1, 2$, and $\tilde{\rho}_1, \tilde{\rho}_2, \tilde{\rho} = \tilde{\rho}_1 \# \tilde{\rho}_2$ be their lifts to $\text{SL}(2, \mathbb{C})$. If we put $s_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, then all region-colors are uniquely determined by

$$s_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad s_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad s_4 = \begin{pmatrix} -1 \\ 3 \end{pmatrix}, \quad s_5 = \begin{pmatrix} -1 \\ 4 \end{pmatrix},$$

$$s_6 = \begin{pmatrix} 4x+3 \\ 4x+7 \end{pmatrix}, \quad s_7 = \begin{pmatrix} 4x+3 \\ 4 \end{pmatrix}, \quad s_8 = \begin{pmatrix} 4x-2 \\ -x-1 \end{pmatrix}, \quad s_9 = \begin{pmatrix} 3x-2 \\ -x-1 \end{pmatrix}.$$

Note that this region-coloring satisfies Lemma 3.1. If we put

$$p = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

then it satisfies (5).

Let $W_1(w_1, \dots, w_5)$ and $W_2(w_4, \dots, w_9)$ be the potential functions of 3_1 and 4_1 from Figure 14, respectively. Then

$$W_1 = \left\{ -\text{Li}_2\left(\frac{w_2}{w_1}\right) - \text{Li}_2\left(\frac{w_2}{w_4}\right) + \text{Li}_2\left(\frac{w_2 w_3}{w_1 w_4}\right) + \text{Li}_2\left(\frac{w_1}{w_3}\right) + \text{Li}_2\left(\frac{w_4}{w_3}\right) + \log \frac{w_1}{w_3} \log \frac{w_4}{w_3} \right\}$$

$$+ \left\{ -\text{Li}_2\left(\frac{w_3}{w_1}\right) - \text{Li}_2\left(\frac{w_3}{w_4}\right) + \text{Li}_2\left(\frac{w_3 w_5}{w_1 w_4}\right) + \text{Li}_2\left(\frac{w_1}{w_5}\right) + \text{Li}_2\left(\frac{w_4}{w_5}\right) + \log \frac{w_1}{w_5} \log \frac{w_4}{w_5} \right\}$$

$$+ \left\{ -\text{Li}_2\left(\frac{w_5}{w_1}\right) - \text{Li}_2\left(\frac{w_5}{w_4}\right) + \text{Li}_2\left(\frac{w_2 w_5}{w_1 w_4}\right) + \text{Li}_2\left(\frac{w_1}{w_2}\right) + \text{Li}_2\left(\frac{w_4}{w_2}\right) + \log \frac{w_1}{w_2} \log \frac{w_4}{w_2} \right\} - \frac{\pi^2}{2},$$

$$\begin{aligned}
W_2 = & \left\{ \operatorname{Li}_2\left(\frac{w_4}{w_5}\right) + \operatorname{Li}_2\left(\frac{w_4}{w_6}\right) - \operatorname{Li}_2\left(\frac{w_4 w_7}{w_5 w_6}\right) - \operatorname{Li}_2\left(\frac{w_5}{w_7}\right) - \operatorname{Li}_2\left(\frac{w_6}{w_7}\right) - \log \frac{w_5}{w_7} \log \frac{w_6}{w_7} \right\} \\
& + \left\{ \operatorname{Li}_2\left(\frac{w_7}{w_6}\right) + \operatorname{Li}_2\left(\frac{w_7}{w_9}\right) - \operatorname{Li}_2\left(\frac{w_4 w_7}{w_6 w_9}\right) - \operatorname{Li}_2\left(\frac{w_6}{w_4}\right) - \operatorname{Li}_2\left(\frac{w_9}{w_4}\right) - \log \frac{w_6}{w_4} \log \frac{w_9}{w_4} \right\} \\
& + \left\{ -\operatorname{Li}_2\left(\frac{w_5}{w_7}\right) - \operatorname{Li}_2\left(\frac{w_5}{w_8}\right) + \operatorname{Li}_2\left(\frac{w_5 w_9}{w_7 w_8}\right) + \operatorname{Li}_2\left(\frac{w_7}{w_9}\right) + \operatorname{Li}_2\left(\frac{w_8}{w_9}\right) + \log \frac{w_7}{w_9} \log \frac{w_8}{w_9} \right\} \\
& + \left\{ -\operatorname{Li}_2\left(\frac{w_9}{w_4}\right) - \operatorname{Li}_2\left(\frac{w_9}{w_8}\right) + \operatorname{Li}_2\left(\frac{w_5 w_9}{w_4 w_8}\right) + \operatorname{Li}_2\left(\frac{w_4}{w_5}\right) + \operatorname{Li}_2\left(\frac{w_8}{w_5}\right) + \log \frac{w_4}{w_5} \log \frac{w_8}{w_5} \right\},
\end{aligned}$$

and the potential function $W(w_1, \dots, w_9)$ of $3_1 \# 4_1$ from Figure 14 is

$$W(w_1, \dots, w_9) = W_1(w_1, \dots, w_5) + W_2(w_4, \dots, w_9).$$

Let

$$\begin{aligned}
\mathcal{I} &:= \left\{ \exp \left(w_k \frac{\partial W}{\partial w_k} \right) = 1 \mid k = 1, \dots, 9 \right\}, \\
\mathcal{I}_1 &:= \left\{ \exp \left(w_k \frac{\partial W_1}{\partial w_k} \right) = 1 \mid k = 1, \dots, 5 \right\}, \\
\mathcal{I}_2 &:= \left\{ \exp \left(w_k \frac{\partial W_2}{\partial w_k} \right) = 1 \mid k = 4, \dots, 9 \right\},
\end{aligned}$$

and define $\mathbf{w}^{(0)} := (w_1^{(0)}, \dots, w_9^{(0)})$ using the formula (7) as follows:

$$\begin{aligned}
w_1^{(0)} &= -3, \quad w_2^{(0)} = -1, \quad w_3^{(0)} = -1, \quad w_4^{(0)} = 5, \quad w_5^{(0)} = 6, \\
w_6^{(0)} &= -4x + 1, \quad w_7^{(0)} = -8x - 2, \quad w_8^{(0)} = -9x + 3, \quad w_9^{(0)} = -7x + 3.
\end{aligned}$$

We put $\mathbf{w}_1^{(0)} = (w_1^{(0)}, \dots, w_5^{(0)})$ and $\mathbf{w}_2^{(0)} = (w_4^{(0)}, \dots, w_9^{(0)})$. Then $\mathbf{w}_1^{(0)}$, $\mathbf{w}_2^{(0)}$ and $\mathbf{w}^{(0)}$ are solutions of \mathcal{I}_1 , \mathcal{I}_2 and \mathcal{I} , respectively. Furthermore, numerical calculation shows

$$\begin{aligned}
i(\operatorname{vol}(\rho_1) + i \operatorname{cs}(\rho_1)) &\equiv (W_1)_0(\mathbf{w}_1^{(0)}) \equiv i(0 + 1.6449\dots i) \pmod{\pi^2}, \\
i(\operatorname{vol}(\rho_2) + i \operatorname{cs}(\rho_2)) &\equiv (W_2)_0(\mathbf{w}_2^{(0)}) \equiv \begin{cases} i(2.0299\dots + 0i) & \text{if } x = \frac{-1-\sqrt{3}i}{2} \\ i(-2.0299\dots + 0i) & \text{if } x = \frac{-1+\sqrt{3}i}{2} \end{cases} \pmod{\pi^2},
\end{aligned}$$

and

$$\begin{aligned}
& i(\operatorname{vol}(\rho_1 \# \rho_2) + i \operatorname{cs}(\rho_1 \# \rho_2)) \equiv W_0(\mathbf{w}^{(0)}) \\
& \equiv \begin{cases} i(2.0299\dots + 1.6449\dots i) & \text{if } x = \frac{-1-\sqrt{3}i}{2} \\ i(-2.0299\dots + 1.6449\dots i) & \text{if } x = \frac{-1+\sqrt{3}i}{2} \end{cases} \\
& \equiv (W_1)_0(\mathbf{w}_1^{(0)}) + (W_2)_0(\mathbf{w}_2^{(0)}) \equiv i(\operatorname{vol}(\rho_1) + i \operatorname{cs}(\rho_1)) + i(\operatorname{vol}(\rho_2) + i \operatorname{cs}(\rho_2)) \pmod{\pi^2},
\end{aligned}$$

which confirms the additivity of the complex volume in Corollary 3.3.

To calculate the twisted Alexander polynomials, we put the Wirtinger presentations of 3_1 , 4_1 and $3_1\#4_1$ from Figure 14 by

$$\begin{aligned}\pi_1(3_1) &= \langle \alpha_1, \alpha_2, \alpha_3 \mid \alpha_1\alpha_2\alpha_1^{-1}\alpha_3^{-1}, \alpha_2\alpha_3\alpha_2^{-1}\alpha_1^{-1}, \alpha_3\alpha_1\alpha_3^{-1}\alpha_2^{-1} \rangle \\ &= \langle \alpha_1, \alpha_2, \alpha_3 \mid \alpha_1\alpha_2\alpha_1^{-1}\alpha_3^{-1}, \alpha_2\alpha_3\alpha_2^{-1}\alpha_1^{-1} \rangle, \\ \pi_1(4_1) &= \langle \alpha_3, \alpha_4, \alpha_5, \alpha_6 \mid \alpha_3\alpha_6\alpha_3^{-1}\alpha_5^{-1}, \alpha_5\alpha_4\alpha_5^{-1}\alpha_3^{-1}, \alpha_6\alpha_4\alpha_6^{-1}\alpha_5^{-1} \rangle, \\ \pi_1(3_1\#4_1) &= \langle \alpha_1, \alpha_2, \alpha_3, \alpha'_3, \alpha_4, \alpha_5, \alpha_6 \mid \alpha_1\alpha_2\alpha_1^{-1}\alpha_3^{-1}, \alpha_2\alpha'_3\alpha_2^{-1}\alpha_1^{-1}, \\ &\quad \alpha'_3\alpha_1(\alpha'_3)^{-1}\alpha_2^{-1}, \alpha_3\alpha_6\alpha_3^{-1}\alpha_5^{-1}, \alpha_5\alpha_4\alpha_5^{-1}\alpha_3^{-1}, \alpha_6\alpha_4\alpha_6^{-1}\alpha_5^{-1} \rangle,\end{aligned}$$

respectively. (If we use Lemma 4.1, the fundamental group $\pi_1(3_1\#4_1)$ can be expressed simply by

$$\begin{aligned}\pi_1(3_1\#4_1) &= \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \mid \alpha_1\alpha_2\alpha_1^{-1}\alpha_3^{-1}, \alpha_2\alpha_3\alpha_2^{-1}\alpha_1^{-1}, \\ &\quad \alpha_3\alpha_6\alpha_3^{-1}\alpha_5^{-1}, \alpha_5\alpha_4\alpha_5^{-1}\alpha_3^{-1}, \alpha_6\alpha_4\alpha_6^{-1}\alpha_5^{-1} \rangle.\end{aligned}$$

This presentation shows (12) trivially, so we are using the Wirtinger presentation of $\pi_1(3_1\#4_1)$ instead.) The Alexander matrices associated to $\tilde{\rho}_1$, $\tilde{\rho}_2$ and $\tilde{\rho}_1\#\tilde{\rho}_2$ obtained by the above Wirtinger presentations are

$$M_{\tilde{\rho}_1} = \begin{pmatrix} 1-t & 0 & 0 & -t & -1 & 0 \\ -t & 1-t & t & 2t & 0 & -1 \\ -1 & 0 & 1 & t & t & -t \\ 0 & -1 & -t & 1-2t & 0 & t \end{pmatrix},$$

$$M_{\tilde{\rho}_2} = \begin{pmatrix} 1+xt & -(x+1)t & 0 & 0 & -1 & 0 & t & 0 \\ (x+1)t & 1-(x+2)t & 0 & 0 & 0 & -1 & t & t \\ -1 & 0 & -xt & (x+1)t & 1-t & 0 & 0 & 0 \\ 0 & -1 & -(x+1)t & (x+2)t & -t & 1-t & 0 & 0 \\ 0 & 0 & t & (x+1)t & -1 & 0 & 1+xt & -(x+1)t \\ 0 & 0 & 0 & t & 0 & -1 & (x+1)t & 1-(x+2)t \end{pmatrix},$$

and $M_{\tilde{\rho}_1\#\tilde{\rho}_2} =$

$$\begin{pmatrix} 1-t & 0 & 0 & -t & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -t & 1-t & t & 2t & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & t & 0 & 0 & t & -t & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -t & 1-2t & 0 & 0 & 0 & t & 0 & 0 & 0 & 0 & 0 & 0 \\ t & 0 & -1 & 0 & 0 & 0 & 1-t & t & 0 & 0 & 0 & 0 & 0 & 0 \\ t & t & 0 & -1 & 0 & 0 & 0 & 1-t & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+xt & -(x+1)t & 0 & 0 & 0 & 0 & -1 & 0 & t & 0 \\ 0 & 0 & 0 & 0 & (x+1)t & 1-(x+2)t & 0 & 0 & 0 & 0 & 0 & -1 & t & t \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -xt & (x+1)t & 1-t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -(x+1)t & (x+2)t & -t & 1-t & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t & (x+1)t & -1 & 0 & 1+xt & -(x+1)t \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t & 0 & -1 & (x+1)t & 1-(x+2)t \end{pmatrix},$$

respectively. The corresponding twisted Alexander polynomials obtained by (10) are

$$\begin{aligned}\Delta_{3_1, \tilde{\rho}_1}(t) &= 1+t^2, \\ \Delta_{4_1, \tilde{\rho}_2}(t) &= t^2(1-4t+t^2), \\ \Delta_{3_1\#4_1, \tilde{\rho}_1\#\tilde{\rho}_2}(t) &= (1-t)^2(1+t^2)t^2(1-4t+t^2),\end{aligned}$$

respectively.⁴ Therefore, we obtain

$$\Delta_{3_1 \# 4_1, \tilde{\rho}_1 \# \tilde{\rho}_2}(t) = (1-t)^2 \Delta_{3_1, \tilde{\rho}_1}(t) \Delta_{4_1, \tilde{\rho}_2}(t), \quad (13)$$

which confirms Corollary 4.2.

A Errata

This appendix is the errata of this article. (The author appreciates Seonhwa Kim for pointing out the error.) The author found the errors after the publication, so he wrote this errata and submitted it to the same journal again. He sincerely apologizes to the readers for confusing them.

For given boundary-parabolic representations $\rho_j : \pi_1(K_j) \rightarrow \text{PSL}(2, \mathbb{C})$ ($j = 1, 2$), the connected sum $\rho_1 \# \rho_2 : \pi_1(K_1 \# K_2) \rightarrow \text{PSL}(2, \mathbb{C})$ was defined at this article. He proved that $\rho_1 \# \rho_2$ is well-defined up to conjugation at Theorem 2.2, but the statement and the proof are not correct.

As an counterexample of Theorem 2.2, consider the example of Fig 14 in Section 5. We put

$$\begin{aligned} a_1 &= \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}, \quad a_3 = \begin{pmatrix} 0 & \\ & 1 \end{pmatrix} = a'_3, \\ a_4 &= \begin{pmatrix} x+1 & \\ & x \end{pmatrix}, \quad a_5 = \begin{pmatrix} x & \\ & x \end{pmatrix}, \quad a_6 = \begin{pmatrix} x & \\ & 0 \end{pmatrix}, \end{aligned} \quad (14)$$

but we can conjugate the figure-eight knot part by the map $*a_3 : \mathcal{P} \rightarrow \mathcal{P}$. The changed arc-colors are

$$\begin{aligned} a_1 &= \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}, \quad a_3 = \begin{pmatrix} 0 & \\ & 1 \end{pmatrix} = \begin{pmatrix} 0 & \\ & 1 \end{pmatrix} * \begin{pmatrix} 0 & \\ & 1 \end{pmatrix} = a'_3, \\ a_4 &= \begin{pmatrix} x+1 & \\ & x \end{pmatrix} * \begin{pmatrix} 0 & \\ & 1 \end{pmatrix} = \begin{pmatrix} x+1 & \\ & 2x+1 \end{pmatrix}, \quad a_5 = \begin{pmatrix} x & \\ & x \end{pmatrix} * \begin{pmatrix} 0 & \\ & 1 \end{pmatrix} = \begin{pmatrix} x & \\ & 2x \end{pmatrix}, \\ a_6 &= \begin{pmatrix} x & \\ & 0 \end{pmatrix} * \begin{pmatrix} 0 & \\ & 1 \end{pmatrix} = \begin{pmatrix} x & \\ & x \end{pmatrix}. \end{aligned} \quad (15)$$

The representations defined by (14) and (15) cannot be conjugate, so the connected sum cannot be well-defined.

The error lies in the third sentence of the proof of Theorem 2.2: “For any $b \in \mathcal{P}$, there exists unique $c \in \mathcal{P}$ such that $b * c = a$.” The author confused that the bijectiveness of the map $*c$ implies this statement. This statement is wrong, so all of the proof is wrong. (For example, in Fig 8, the second diagram is wrong. We cannot guarantee the arc-color of the small box becomes $D_1 * c$.) Therefore, Theorem 2.2 is wrong and Definition 2.1 should be modified.

⁴To calculate the determinants, final two columns of all three matrices are removed.

One way to solve these errors is to consider the connected sum $\rho_1 \# \rho_2$ not as a *definition*, but a method to construct boundary-parabolic representations. This construction does not define the unique representation, but it defines many representations. Under this construction, Proposition 2.3 should be changed as follows.

Proposition A.1 (New version of Proposition 2.3). *For a boundary-parabolic representation $\rho : \pi_1(K_1 \# K_2) \rightarrow \mathrm{PSL}(2, \mathbb{C})$, there exist $\rho_1 : \pi_1(K_1) \rightarrow \mathrm{PSL}(2, \mathbb{C})$ and $\rho_2 : \pi_1(K_2) \rightarrow \mathrm{PSL}(2, \mathbb{C})$, which are unique up to conjugation, such that one of $\rho_1 \# \rho_2$ becomes ρ .*

Proof. The existence is trivial from Fig 9. The uniqueness follows from Fig 7 because the arc-color of D is invariant under the moves up to conjugation. \square

Interestingly, Section 3–4 are still true under this construction. This implies that any representation obtained by $\rho_1 \# \rho_2$ has the same complex volume

$$(\mathrm{vol}(\rho_1) + i \mathrm{cs}(\rho_1)) + (\mathrm{vol}(\rho_2) + i \mathrm{cs}(\rho_2))$$

and the same twisted Alexander polynomial

$$\Delta'_{K_1, \widetilde{\rho_1}} \cdot \Delta'_{K_2, \widetilde{\rho_2}}.$$

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